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Mixed Method for the Product Integral on the Infinite Interval

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ABSTRACT

In this note, quadrature formula is constructed for product integral on the infinite interval $I(f)=\int_a^{\infty} w(x) f(x)dx,$ where *w(x)* is a weight function and $f(x)$ is a smooth decaying function for $x>N$ (large enough) and piecewise discontinuous function of the first kind on the interval $a \le x \le N$. For the approximate method we have reduced infinite interval $x\in[a,\infty)$ into the interval $t\in[0,1]$ and used the mixed method: Cubic Newton's divided difference formula on $[0,t_3]$ and Romberg method on $[t_3,1]$ with equal step size, $t_i=t_0+ih,\ i=0,...,n,\ h=1/n,$ where $t_0=0,\ t_n=1.$ Error term is obtained for mixed method on different classes of functions. Finally, numerical examples are presented to validate the method presented.

Keywords: Product integral, Romberg method, mixed method, error estimate.

1. INTRODUCTION

Integration problems on infinite interval are not defined the same as the finite interval it is usually defined as improper integrals

$$\int_a^{\infty} w(x) f(x) dx = \lim_{N \to \infty} \int_a^{N} w(x) f(x) dx, \tag{1}$$

where $w(x)$ is a weight function and infinite integral (1) exists whenever the latter limit exists. It is known that the continuity of integrant function or boundedness of function $f(x)$ is not enough the existence of the infinite boundary integral (1). Quadrature formula of the type

$$I_n(f) = \sum_{j=0}^{n} w_{j,n} f(x_{j,n}), \tag{2}$$

for the estimate of the product integral

$$I(fw) = \int_a^b w(x) f(x) dx, \tag{3}$$

where a and b are finite numbers and $w(x)$ is a weight function, have been extensively studied in the 1970-2000 (Branders and Piessens, Philip Rabinowitz and Ian Sloan, Philip Rabinowitz, Gonzailez-Vera and Santos-Leon, Dagnino and Rabinowitz and so on) and literature cited therein. Oscillatory integration problems (Hasegawa, Iserles and so on) have been investigated for many decades and recently Kang and Xiang (2013) have proposed new techniques for the numerical evaluation of a class of highly oscillatory integrals containing algebraic singularities. Error and convergence analysis and robust numerical examples are used to demonstrate the accuracy and effectiveness of the proposed method. Unfortunately, few works can be found on the investigation of the product integral in the infinite interval (Simpson, Avram SIDI, Jezequel and Chesneaux, Hascelik). Romberg integration for (3) with $w(x) = 1$ and its application for solving Volterra integral equation on the finite interval have been investigated in Mestrovic and Ocvirk.

There are many techniques on reducing the infinite interval into the finite interval, but reduction technique brings singularity of the integrand function and application of the Romberg rule (Davis and Rabinowitz, Burden and Faires) is not very suitable for those parts where singularity appears. In this note, we consider product integral (1) and reduce it into the interval $[0,1]$. Moreover as an approximation we have used the mixed method: Cubic Newton's divided difference formula on $[t_{n-3},1]$ and Romberg method on $[0,t_{n-3}]$ with equal step size. Error estimate is established for the proposed method in the class of function $f \in C^4[0,1]$. Finally, two examples are provided to demonstrate the efficiency of the method proposed.

2. DERIVATION OF THE APPROXIMATE METHOD (CHANGE OF VARIBLE)

The trapezoidal rule is one of the simplest of the integration formulas, but it is not always sufficiently accurate. Thus, the Romberg method uses the composite Trapezoidal rule to give preliminary approximations, and then applies Richardson extrapolation to obtain improved approximations.

Recall the ordinary integration problem

$$I(f) = \int_a^b f(x)dx \tag{4}$$

Definition 1. (Burden and Faires, pp. 207): Composite trapezoidal approximation for (4) of a function f on the interval $[a,b]$ is

$$R_{k,1} = \frac{1}{2}\left[R_{k-1,1} + h_{k-1} \sum_{i=1}^{2^{k-2}} f\left(a + (2i-1)h_k\right) \right]$$

where

$$h_k = \frac{b-a}{m_k} = \frac{b-a}{2^{k-1}}, \text{ and } R_{1,1} = \frac{b-a}{2}\left[f(a) + f(b) \right]$$

for each $k = 2,3,...,n.$.

Definition 2 (Burden and Faires (2005)): The Romberg integration rule for each $k = 2,3,\ldots,n$ and $j = 2,3,\ldots,k$ is defined as

$$R_{k,j} = R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1}-1} = \frac{1}{4^{j-1}-1}\left(4^{j-1}R_{k,j-1} - R_{k-1,j-1}\right) \tag{5}$$

Theorem 1. (Davis and Rabinowitz (1984)): Let $f \in C^{2j+2}[0,1]$ be a real function to be integrated over $[a,b]$ and $R_{k,j}$ be defined in Romberg's method (5), then remainder term $R_{k,j}$ is zero for $f \in P_{2j+1}$, and truncation error of $R_{k,j}$ is given by

$$E(f) = \left| R_{k,j} - \int_0^1 f(t)dt \right| \le r_j h_k^{2j} \left| f^{(2j+2)}(\xi_t) \right|,$$

where $r_j = \dfrac{2^{-j(j-1)} \left| B_{2j+2} \right|}{(2j+2)!}$, with Bernoulli numbers B_{2j+2}.

The Romberg technique has the desirable feature that it allows an entire new row in the table to be calculated by doing only one additional application of the Composite Trapezoidal rule. It then uses a simple averaging on the previously calculated values to obtain the remaining entries in the row. Moreover, the Romberg method has the advantage that all of the weights w_n are positive and the abscissas x_i are equally spaced.

The development of Romberg integration relies on the theoretical assumption that $f(x)$ is smooth enough so that the error in the trapezoidal rule can be expanded in a series involving only even powers of h.

2.1. Change of Variable

Due to the reduction technique, most of the infinite integral will be reduced to [0, 1] or [-1,1] interval. The substitution

$$t = e^{1-\frac{x}{a}}, \quad x \ge a$$

in the product integral (1) yields

$$I(fw) = \int_a^\infty w(x) f(x) dx = a \int_0^1 w_1(t) g_1(t) dt . \qquad (6)$$

where $w_1(t) = w\left(a \ln \dfrac{e}{t}\right), \quad g_1(t) = \dfrac{1}{t} f\left(a \ln \dfrac{e}{t}\right).$

If density function f in (6) is smooth enough and tends to zero at infinite (∞) faster than the denominator t of the fraction function $g_1(t)$, i.e.

$$\lim_{t\to 0} g_1(t) = \lim_{t\to 0} \frac{1}{t} f\left(a \ln \frac{e}{t}\right) = 0.$$

then product integral (6) is well defined.

Since Romberg method is improvement of Trapezoidal rule and therefore it might not work well when density function has a singularity in the domain of integration. Fortunately, spline approximation is good for singularity problems. For mixed method, we will split the interval $[0,1]$ into $[0,t_3]$ and $[t_3,1]$, where $t_k = kh,\ k = 0,1,\ldots,n.$

$$\begin{aligned} I(wf) &= a\int_0^1 w_1(t)\, g_1(t)\, dt = a\left(\int_0^{t_3} + \int_{t_3}^1\right) w_1(t)\, g_1(t)\, dt \\ &= a\left[I_1(wf) + I_2(wf)\right], \end{aligned} \tag{7}$$

where

$$I_1(wf) = \int_0^{t_3} w_1(t) g_1(t)\, dt, \qquad I_2(wf) = \int_{t_3}^1 w_1(t) g_1(t)\, dt.$$

The first integral is approximated with modified cubic Newton polynomial $L_3(x)$, while the second integral is approximated by modified Romberg method

2.2. Construction of Quadrature Formula

Let $L_3(x)$ be defined by the 3-rd order Newton Divided Difference formula i.e.

$$\begin{aligned} L_3(t) = {} & f(t_0) + f(t_0,t_1)(t-t_0) + f(t_0,t_1,t_2)(t-t_0)(t-t_1) \\ & + f(t_0,t_1,t_2,t_3)(t-t_0)(t-t_1)(t-t_2) \end{aligned}$$

or Modified Newton Divided Difference formula for the function $g_1(x)$ is

$$
\begin{aligned}
L_3(t) = &-\frac{1}{6h^3}(t-t_1)(t-t_2)(t-t_3)g_1(t_0)+\frac{1}{2h^3}(t-t_0)(t-t_2)(t-t_3)g_1(t_1) \\
&-\frac{1}{2h^3}(t-t_0)(t-t_1)(t-t_3)g_1(t_2)+\frac{1}{6h^3}(t-t_0)(t-t_1)(t-t_2)g_1(t_3)
\end{aligned} \tag{8}
$$

Let us consider the first integral in (7).

$$
\begin{aligned}
I_1(wf) &\cong I_1(wL_3) = \int_0^{t_3} w_1(t)L_3(t)dt \\
&= \int_0^{t_3} w_1(t)\Big[-\frac{1}{6h^3}(t-t_1)(t-t_2)(t-t_3)g_1(t_0)+\frac{1}{2h^3}(t-t_0)(t-t_2)(t-t_3)g_1(t_1) \\
&\quad -\frac{1}{2h^3}(t-t_0)(t-t_1)(t-t_3)g_1(t_2)+\frac{1}{6h^3}(t-t_0)(t-t_1)(t-t_2)g_1(t_3)dt\Big] \\
&= \sum_{k=0}^{3} A_k(h)g_1(t_k),
\end{aligned} \tag{9}
$$

where

$$
A_0(h) = -\frac{1}{6h^3}\int_0^{t_3} w_1(t)(t-t_1)(t-t_2)(t-t_3)dt,
$$

$$
A_1(h) = \frac{1}{2h^3}\int_0^{t_3} w_1(t)(t-t_0)(t-t_2)(t-t_3)dt,
$$

$$
A_2(h) = -\frac{1}{2h^3}\int_0^{t_3} w_1(t)(t-t_0)(t-t_1)(t-t_3)dt,
$$

$$
A_3(h) = \frac{1}{6h^3}\int_0^{t_3} w_1(t)(t-t_0)(t-t_1)(t-t_2)dt.
$$

For the second integral in Eq. (7) the following substitution (Gonzailes-Vera and Santos-Leon (1996))

$$
\begin{aligned}
&y = H(t) = \frac{1}{c}\int_{t_3}^{t} w_1(\tau)d\tau, \quad 0 < c < \infty, \\
&t = L(y) = H^{-1}(t), \quad t \in [t_3, 1]
\end{aligned} \tag{10}
$$

is used to construct quadrature formula. Additionally here and hereafter, $w_1(t)$ is assumed to satisfy $w_1(t) \geq 0$ on [0,1] with $\int_0^1 w_1(t)dt = c < \infty$, and

$w_1(t)$ does not identically vanish on any subinterval of $[0,1]$. So that for the integral $I_2(wf)$ in Eq. (7) we have the following trapezoidal formula

$$I_2\left(wf\right)=\int_{t_3}^{1} w_1(t)g_1(t)dt=\int_{0}^{1} g_1(L(y))dy \cong \frac{1}{n}\left[g_{10}+g_{1N}+2\sum_{l+1}^{n-1} g_{1l}\right],$$

where $g_{1l}=g_1(L(y_1)),\ l=0,1,...,n$.

Let

$$T_{k,0}=h_k\sum_{j=2}^{2^k}{}'' g_1(h_k j),\ h_k=2^{-k},\ k\geq 1,$$

be trapezoidal sums on the interval $[0,1]$. The double prime indicates that the first and last term are to be multiplied by $\frac{1}{2}$.

Now we can use Romberg integration rule which can be written as

$$T_{k,m}=T_{k,m-1}+\frac{T_{k,m-1}-T_{k-1,m-1}}{4^{m-1}-1}=\frac{1}{4^{m-1}-1}\left(4^{m-1}T_{k,m-1}-T_{k-1,m-1}\right). \qquad (11)$$

where $k=1,2,...,n$ and $m=1,2,...,k$. Thus quadrature formula for product integral (7) is calculated by (9) and (11).

3. CONVERGENCE ANALYSIS

It is known from the theory of Lagrange interpolation formula that the errors estimate of interpolation polynomials depends on the smoothness of the integrant function.

Theorem 2 (Burden and Faires (2005)): Let $f\in C^{(n+1)}[a,b]$, then the error estimate of Lagrange interpolation formula is

$$R_n(x)=f(x)-L_n(x)=\frac{f^{(n+1)}\left(\xi_x\right)}{(n+1)!}\left(x-x_0\right)...\left(x-x_n\right)$$

Particularly, for the function $g_1(t)$ with $n=3$ and interval $t\in[0,t_3]$ we have

$$R_3(t)=g_1(t)-L_3(t)=\frac{h^4\cdot g_1^{(4)}(\xi_t)}{4!}\tau(\tau-1)(\tau-2)(\tau-3), \qquad (12)$$

where

$$\tau=\frac{t-t_0}{h}, \quad h=\frac{1}{n}, \quad t_0=0\,.$$

Hence, we will proof the following main theorem.

Theorem 3: Let $f\in C^4[0,1]$, and $T_{k,3}$ be defined in Romberg's method (10), then remainder term $E(wf)$ is zero for all $f\in P_4$ and truncation error of $E(wf)$ has the form

$$E(wf)=\left|I(wf)-(I_1(wL_3)+T_{k,3})\right|\le O(h^5).$$

Proof.

We write

$$E(wf)=\left|\int_0^1 w_1(t)g_1(t)dt-\left(I_1(wL_3)+T_{k,3}\right)\right|$$
$$\le\left|\int_0^{t_3} w_1(t)\left[g_1(t)-L_3(t)\right]dt\right|+\left|\int_0^1 g_1(L(y))dy-T_{k,3}\right|$$

Due to Theorem 1 and 2 and Eq. (11) yields

$$E_n(f)\le\|g_1-P_2\|\|w_1\|3h+h_k^6|r_3|\left\|g_1^{(6)}\right\|=O\left(h^5\right)+O\left(h^6\right)=O\left(h^5\right).$$

where the norm $\|\bullet\|$ is the Chebyshev norm. Theorem 3 is proven.

In this section, we have shown that the proposed mixed method is to be uniformly convergent on the infinite interval to the to the integral with the minimum order $O(h^5)$. It shows that mixed method has a higher accurate approximation method and it decreases very fast by increasing the number of points.

4. NUMERICAL EXAMPLES

Example 1: Let us consider the following product integral.

$$Q_1 = \int_0^\infty 10e^{-2x}(x^2+1)dx = 10\int_0^1 y[(\ln y)^2+1]dy\theta.$$

Solution: Exact solution of the product integral Q_1 is $Q_1 = 7.5.$ This product integral has a singularity at the point "0" but this singularity is removable singularity. Numerical results and comparisons are given in Table 1.

TABLE 1: Comparison of the Mixed method with Romberg method for problem Q_1

N	C	Exact value	Romberg method	Mixed method	Error 1 (Exact and Romberg)	Error 2 (Exact and Mixed Method)
16	1	7.50	7.429370880	7.467471252	0.070629120	0.032528748
	2	7.50	7.473458648	7.466256983	0.026541352	0.013743017
32	1	7.50	7.476584315	7.489618657	0.023415685	0.010381343
	2	7.50	7.492322326	7.490176929	0.007677674	0.003823071
64	1	7.50	7.492510676	7.496606434	0.007489324	0.003393566
	2	7.50	7.497819662	7.497197233	0.002180338	0.001802767
256	1	7.50	7.499290705	7.499647302	0.000709295	0.000352698
	2	7.50	7.499831319	7.499781475	0.000168681	0.000118525

N = number of points, ***C***=number of column in Romberg's method.

Example 2: Find an approximate solution of the product integral.

$$Q_2 = \int_0^\infty x^{-\frac{3}{2}} \sin\left(\frac{1}{x}\right)dx = \int_0^1 \frac{\sin t}{\sqrt{t}}dt$$

Solution: This product integral has a singularity at the end point "0" and at this singular point integrant function tends to zero. Therefore we can use Romberg method. The exact solution is $Q_2 = 0.6205366.$

TABLE 2: Comparison of the Mixed method with Romberg method for problem Q_2

N	C	Exact value	Romberg method	Mixed method	Error 1 (Exact and Romberg)	Error 2 (Exact and Mixed method)
16	1	0.62053661	0.61732721	0.61878139	0.00320938	0.00175515
	2	0.62053661	0.61926835	0.61899556	0.00126824	0.00114104
32	1	0.62053661	0.61939787	0.61989174	0.00113872	0.00064486
	2	0.62053661	0.62008810	0.61999152	0.00044849	0.00034508
64	1	0.62053661	0.62013298	0.62030369	0.00040361	0.00023291
	2	0.62053661	0.62037801	0.62034384	0.00015858	0.00011276
256	1	0.62053661	0.62048602	0.62050695	0.00005057	0.00002965
	2	0.62053661	0.62051683	0.62051250	0.00001976	0.00001410

N = number of points, ***C***=number of column in Romberg's method

Tables 1-2 show the comparison results between proposed mixed approximation method and classical Romberg method for different number of points. Both methods are convergent to the exact solution when the integrant function has removable singularity. Generally, the computational results show that the proposed method is performed slightly better when compare to that standard Romberg method.

5. CONCLUSION

In this paper, we have constructed quadrature formula for product integral in the infinite interval. Mixed method is used to get approximate solution. Theorem 2 shows that mixed method is exact for the polynomial of degree 4 and order of convergence is at least 5. Many new results can be obtained from the approximation of the mixed method. From Table 1 and Table 2, we can conclude that the mixed method is more accurate compared to that classical Romberg method which is very reliable for product integral.

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